

ON THE NUMBER OF COLLINEAR TRIPLES IN PERMUTATIONS

LIANGPAN LI

ABSTRACT. Let $\alpha : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ be a permutation and $\Psi(\alpha)$ be the number of collinear triples modulo n in the graph of α . Cooper and Solymosi had given by induction the bound $\min_{\alpha} \Psi(\alpha) \geq \lceil (n-1)/4 \rceil$ when n is a prime number. The main purpose of this paper is to give a direct proof of that bound. Besides, the expected number of collinear triples a permutation can have is also been determined.

1. INTRODUCTION

Let $\alpha : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ be a permutation and $\Psi(\alpha)$ be the number of collinear triples modulo n in

$$\Gamma(\alpha) = \{(i, \alpha(i)) : i \in \mathbb{Z}_n\},$$

the graph of α . Cooper and Solymosi [1] had given by induction the bound

$$\min_{\alpha} \Psi(\alpha) \geq \lceil (n-1)/4 \rceil$$

when n is a prime number. The main purpose of this paper is to give a direct proof of that bound. Besides, the expected number of collinear triples a permutation can have is also been determined.

It should be noted that Cooper and Solymosi [1] conjectured

$$\min_{\alpha} \Psi(\alpha) = (n-1)/2,$$

and Cooper had improved the above lower bound estimate in a recent preprint [2]. As for the classical no-three-in-line without modulo n problem, we suggest the interested reader visiting Wikipedia for a pleasant exposition.

2. EXPECTED NUMBER OF COLLINEAR TRIPLES

Let $n \geq 3$ be a fixed prime number throughout this paper. Since there are $n!$ permutations in total, the expected number of collinear triples a permutation can have is

$$\mathcal{E}(n) = \frac{\sum_{\alpha} \Psi(\alpha)}{n!}.$$

Choose arbitrarily three different points i, j, k from \mathbb{Z}_n . Since n is a prime number, the possible choices of

$$(\alpha(i), \alpha(j), \alpha(k))$$

in $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ such that

$$(i, \alpha(i)), (j, \alpha(j)), (k, \alpha(k))$$

2000 *Mathematics Subject Classification.* 51E15, 11T99.

Key words and phrases. finite field, collinear triple.

are collinear is $P(n, 2)$. Hence there are $P(n, 2) \cdot (n-3)!$ permutations which have collinear triple at points i, j, k . Consequently

$$\mathcal{E}(n) = \frac{C(n, 3) \cdot P(n, 2) \cdot (n-3)!}{n!} = \frac{n(n-1)}{6}.$$

3. LOWER BOUND ESTIMATES

Let $\alpha : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ be a permutation. Note that for every pair of points in $\Gamma(\alpha)$, the slope of that pair must be in $1, 2, \dots, n-1$. Partition the $C(n, 2)$ pairs into classes $\{S_k\}_{k=1}^{n-1}$ according to their slopes, say for example, every pair in S_k has slope k . The average pairs one class can have is

$$\frac{C(n, 2)}{n-1} = \frac{n}{2}.$$

Let B_k ($1 \leq k \leq n-1$) be the integer satisfying

$$\#S_k = \frac{n}{2} - 0.5 + B_k.$$

Since

$$\sum_{k=1}^{n-1} \#S_k = C(n, 2),$$

it follows that

$$\sum_{k=1}^{n-1} B_k = \frac{n-1}{2},$$

and consequently

$$(3.1) \quad \sum_{k: B_k > 0} B_k \geq \frac{n-1}{2}.$$

Next suppose B_k is a positive integer. Partition $\mathbb{Z}_n \times \mathbb{Z}_n$ into n distinct parallel lines $\{L_s\}_{s=1}^n$ with common slope k . Partition S_k into classes $\{E_s\}_{s=1}^n$ according to the lines the pairs lie in. Write

$$V_s = \#(L_s \cap \Gamma(\alpha)).$$

It is not hard to see that

$$(3.2) \quad \sum_{s=1}^n V_s = n,$$

$$(3.3) \quad \sum_{s=1}^n C(V_s, 2) = \sum_{s=1}^n \#E_s = \#S_k,$$

and there are in total

$$(3.4) \quad \sum_{s=1}^n C(V_s, 3)$$

collinear triples with slope k . For $i = 1, 2, \dots, n$, let

$$m_i = \#\{s : V_s = i\},$$

the number of lines such that every line intersect with $\Gamma(\alpha)$ at exactly i points. With these notations we rewrite (3.2) and (3.3) as

$$(3.5) \quad \sum_{i=1}^n im_i = n,$$

$$(3.6) \quad \sum_{i=2}^n m_i C(i, 2) = \frac{n}{2} - 0.5 + B_k.$$

Multiplying (3.6) by 2 and subtracting it from (3.5) subsequently yields

$$\sum_{i=3}^n (2C(i, 2) - i)m_i = 2B_k - 1 + m_1.$$

Now it is true time for us to estimate from below the number of collinear triples with slope k . According to (3.4) there are

$$(3.7) \quad \sum_{i=3}^n m_i C(i, 3)$$

collinear triples with slope k . Since

$$\min_{3 \leq i \leq n} \frac{C(i, 3)}{2C(i, 2) - i} = \min_{3 \leq i \leq n} \frac{i - 1}{6} = \frac{1}{3},$$

it follows that

$$(3.8) \quad \sum_{i=3}^n m_i C(i, 3) \geq \frac{1}{3} \cdot \sum_{i=3}^n (2C(i, 2) - i)m_i \geq \frac{2B_k - 1 + m_1}{3} \geq \frac{2B_k - 1}{3}.$$

Considering the LHS of (3.8) is an integer, we improve the above estimate slightly into

$$(3.9) \quad \sum_{i=3}^n m_i C(i, 3) \geq \left\lceil \frac{2B_k - 1}{3} \right\rceil \geq \left\lceil \frac{B_k}{2} \right\rceil.$$

Combining (3.9) with (3.1) yields

$$\sum_{k: B_k > 0} \left\lceil \frac{B_k}{2} \right\rceil \geq \left\lceil \sum_{k: B_k > 0} \frac{B_k}{2} \right\rceil \geq \left\lceil \frac{n - 1}{4} \right\rceil.$$

Hence there are at least $\left\lceil \frac{n-1}{4} \right\rceil$ collinear triples in the graph of α .

4. REMARKS

Suppose Γ is a subset of $\mathbb{Z}_n \times \mathbb{Z}_n$ and let $\Psi(\Gamma)$ be the number of collinear triples modulo n in Γ . Cooper and Solymosi [1] proved that if $\#\Gamma \geq n + 3$, then $\Psi(\Gamma) > 0$. In fact, both Cooper and Solymosi' and this paper's methods can further show that

$$(4.1) \quad \#\Gamma = n + 2 \Rightarrow \Psi(\Gamma) \geq \left\lceil \frac{n + 1}{4} \right\rceil,$$

the interested reader can easily provide the details. We conclude this paper with two examples to show that (4.1) is best possible in two senses when $n = 5$. Define two sets in $\mathbb{Z}_5 \times \mathbb{Z}_5$ by

$$\begin{aligned}\Gamma_1 &= \{(0, 0), (0, 1), (1, 2), (1, 3), (2, 2), (4, 1)\}, \\ \Gamma_2 &= \{(0, 0), (0, 1), (1, 2), (1, 3), (2, 2), (4, 1), (2, 1)\}.\end{aligned}$$

Then Γ_1 is free of collinear triples and $\Psi(\Gamma_2) = 2$.

REFERENCES

- [1] J. N. Cooper and J. Solymosi, Collinear points in permutations, *Ann. of Combinatorics* **9** (2005) 169–175.
- [2] J. Cooper, Collinear triples hypergraphs and the finite plane Kakeya problem, *math.CO/0607734*

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAOTONG UNIVERSITY, SHANGHAI 200240, PEOPLE'S
REPUBLIC OF CHINA

E-mail address: liliangpan@yahoo.com.cn